

# ON CLOSURES IN SEMITOPOLOGICAL INVERSE SEMIGROUPS WITH CONTINUOUS INVERSION

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**ABSTRACT.** We study the closures of subgroups, semilattices and different kinds of semigroup extensions in semitopological inverse semigroups with continuous inversion. In particular we show that a topological group  $G$  is  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion if and only if  $G$  is compact, a Hausdorff linearly ordered topological semilattice  $E$  is  $H$ -closed in the class of semitopological semilattices if and only if  $E$  is  $H$ -closed in the class of topological semilattices, and a topological Brandt  $\lambda^0$ -extension of  $S$  is (absolutely)  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is  $S$ . Also, we construct an example of an  $H$ -closed non-absolutely  $H$ -closed semitopological semilattice in the class of semitopological semilattices.

## 1. INTRODUCTION AND PRELIMINARIES

We shall follow the terminology of [2, 8, 12, 27, 30].

A subset  $A$  of an infinite set  $X$  is called *cofinite in  $X$*  if  $X \setminus A$  is finite.

Given a semigroup  $S$ , we shall denote the set of idempotents of  $S$  by  $E(S)$ . A *semilattice* is a commutative semigroup of idempotents. For a semilattice  $E$  the semilattice operation on  $E$  determines the partial order  $\leq$  on  $E$ :

$$e \leq f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called *natural*. An element  $e$  of a partially ordered set  $X$  is called *minimal* if  $f \leq e$  implies  $f = e$  for  $f \in X$ . An idempotent  $e$  of a semigroup  $S$  without zero (with zero  $0_S$ ) is called *primitive* if  $e$  is a minimal element in  $E(S)$  (in  $(E(S)) \setminus \{0_S\}$ ). A *maximal chain* of a semilattice  $E$  is a chain which is properly contained in no other chain of  $E$ . The Axiom of Choice implies the existence of maximal chains in any partially ordered set.

A semigroup  $S$  with the adjoined unit [zero] will be denoted by  $S^1$  [ $S^0$ ] (cf. [8]). Next, we shall denote the unit (identity) and the zero of a semigroup  $S$  by  $1_S$  and  $0_S$ , respectively. Given a subset  $A$  of a semigroup  $S$ , we shall denote by  $A^* = A \setminus \{0_S\}$  and  $|A|$  = the cardinality of  $A$ . A semigroup  $S$  is called *inverse* if for any  $x \in S$  there exists a unique  $y \in S$  such that  $xyx = x$  and  $yxy = y$ . Such an element  $y$  is called *inverse* of  $x$  and it is denoted by  $x^{-1}$ .

If  $h: S \rightarrow T$  is a homomorphism (or a map) from a semigroup  $S$  into a semigroup  $T$  and if  $s \in S$ , then we denote the image of  $s$  under  $h$  by  $(s)h$ . A semigroup homomorphism  $h: S \rightarrow T$  is called *annihilating* if  $(s)h = (t)h$  for all  $s, t \in S$ .

Let  $S$  be a semigroup with zero and  $\lambda$  a cardinal  $\geq 1$ . We define the semigroup operation on the set  $B_\lambda(S) = (\lambda \times S \times \lambda) \cup \{0\}$  as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and  $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$ , for all  $\alpha, \beta, \gamma, \delta \in \lambda$  and  $a, b \in S$ . If  $S = S^1$  then the semigroup  $B_\lambda(S)$  is called the *Brandt  $\lambda$ -extension of the semigroup  $S$*  [13]. Obviously, if  $S$  has zero then

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$\mathcal{J} = \{0\} \cup \{(\alpha, 0_S, \beta) \mid 0_S \text{ is the zero of } S\}$  is an ideal of  $B_\lambda(S)$ . We put  $B_\lambda^0(S) = B_\lambda(S)/\mathcal{J}$  and the semigroup  $B_\lambda^0(S)$  is called the *Brandt  $\lambda^0$ -extension of the semigroup  $S$  with zero* [19].

Next, if  $A \subseteq S$  then we shall denote  $A_{\alpha\beta} = \{(\alpha, s, \beta) \mid s \in A\}$  if  $A$  does not contain zero, and  $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A \setminus \{0\}\} \cup \{0\}$  if  $0 \in A$ , for  $\alpha, \beta \in \lambda$ .

We shall denote the semigroup of  $\lambda \times \lambda$ -matrix units by  $B_\lambda$  and the subsemigroup of  $\lambda \times \lambda$ -matrix units of the Brandt  $\lambda^0$ -extension of a monoid  $S$  with zero by  $B_\lambda^0(1)$ . We always consider the Brandt  $\lambda^0$ -extension only of a monoid with zero. Obviously, for any monoid  $S$  with zero we have  $B_1^0(S) = S$ . Note that every Brandt  $\lambda$ -extension of a group  $G$  is isomorphic to the Brandt  $\lambda^0$ -extension of the group  $G^0$  with adjoined zero. The Brandt  $\lambda^0$ -extension of the group with adjoined zero is called a *Brandt semigroup* [8, 27]. A semigroup  $S$  is a Brandt semigroup if and only if  $S$  is a completely 0-simple inverse semigroup [7, 25] (cf. also [27, Theorem II.3.5]). We also observe that the semigroup  $B_\lambda$  of  $\lambda \times \lambda$ -matrix units is isomorphic to the Brandt  $\lambda^0$ -extension of the two-element monoid with zero  $S = \{1_S, 0_S\}$  and the trivial semigroup  $S$  (i. e.  $S$  is a singleton set) is isomorphic to the Brandt  $\lambda^0$ -extension of  $S$  for every cardinal  $\lambda \geq 1$ .

Let  $\{S_\iota : \iota \in \mathcal{I}\}$  be a disjoint family of semigroups with zero such that  $0_\iota$  is zero in  $S_\iota$  for any  $\iota \in \mathcal{I}$ . We put  $S = \{0\} \cup \bigcup \{S_\iota^* : \iota \in \mathcal{I}\}$ , where  $0 \notin \bigcup \{S_\iota^* : \iota \in \mathcal{I}\}$ , and define a semigroup operation “ $\cdot$ ” on  $S$  in the following way

$$s \cdot t = \begin{cases} st, & \text{if } st \in S_\iota^* \text{ for some } \iota \in \mathcal{I}; \\ 0, & \text{otherwise.} \end{cases}$$

The semigroup  $S$  with the operation “ $\cdot$ ” is called an *orthogonal sum* of the semigroups  $\{S_\iota : \iota \in \mathcal{I}\}$  and in this case we shall write  $S = \sum_{\iota \in \mathcal{I}} S_\iota$ .

A non-trivial inverse semigroup is called a *primitive inverse semigroup* if all its non-zero idempotents are primitive [27]. A semigroup  $S$  is a primitive inverse semigroup if and only if  $S$  is an orthogonal sum of Brandt semigroups [27, Theorem II.4.3].

In this paper all topological spaces are Hausdorff. If  $Y$  is a subspace of a topological space  $X$  and  $A \subseteq Y$ , then by  $\text{cl}_Y(A)$  we denote the topological closure of  $A$  in  $Y$ .

A (semi)topological semigroup is a Hausdorff topological space with a (separately) continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an *inverse topological semigroup*. A *topological inverse semigroup* is an inverse topological semigroup with continuous inversion. We observe that the inversion on a (semi)topological inverse semigroup is a homeomorphism (see [10, Proposition II.1]). A *semitopological group* is a Hausdorff topological space with a separately continuous group operation. A semitopological group with continuous inversion is a *quasitopological group*. A *paratopological group* is called a group with a continuous group operation. A paratopological group with continuous inversion is a *topological group*.

Let  $\mathfrak{STSG}_0$  be a class of semitopological semigroups. A semigroup  $S \in \mathfrak{STSG}_0$  is called *H-closed in  $\mathfrak{STSG}_0$* , if  $S$  is a closed subsemigroup of any topological semigroup  $T \in \mathfrak{STSG}_0$  which contains  $S$  both as a subsemigroup and as a topological space. The *H-closed topological semigroups* were introduced by Stepp in [31], and there they were called *maximal semigroups*. A semitopological semigroup  $S \in \mathfrak{STSG}_0$  is called *absolutely H-closed in the class  $\mathfrak{STSG}_0$* , if any continuous homomorphic image of  $S$  into  $T \in \mathfrak{STSG}_0$  is *H-closed in  $\mathfrak{STSG}_0$* . An algebraic semigroup  $S$  is called:

- *algebraically complete in  $\mathfrak{STSG}_0$* , if  $S$  with any Hausdorff topology  $\tau$  such that  $(S, \tau) \in \mathfrak{STSG}_0$  is *H-closed in  $\mathfrak{STSG}_0$* ;
- *algebraically h-complete in  $\mathfrak{STSG}_0$* , if  $S$  with discrete topology  $\mathfrak{d}$  is absolutely *H-closed in  $\mathfrak{STSG}_0$*  and  $(S, \mathfrak{d}) \in \mathfrak{STSG}_0$ .

Absolutely *H-closed* topological semigroups and algebraically *h-complete* semigroups were introduced by Stepp in [32], and there they were called *absolutely maximal* and *algebraic maximal*, respectively.

Recall [1], a topological group  $G$  is called *absolutely closed* if  $G$  is a closed subgroup of any topological group which contains  $G$  as a subgroup. In our terminology such topological groups are called *H-closed* in the class of topological groups. In [28] Raikov proved that a topological group  $G$  is absolutely closed if and only if it is Raikov complete, i.e.  $G$  is complete with respect to the two-sided uniformity. A

topological group  $G$  is called  *$h$ -complete* if for every continuous homomorphism  $h: G \rightarrow H$  the subgroup  $f(G)$  of  $H$  is closed [9]. In our terminology such topological groups are called absolutely  $H$ -closed in the class of topological groups. The  $h$ -completeness is preserved under taking products and closed central subgroups [9].  $H$ -closed paratopological and topological groups in the class of paratopological groups studied in [29].

In [32] Stepp studied  $H$ -closed topological semilattice in the class of topological semigroups. There he proved that an algebraic semilattice  $E$  is algebraically  $h$ -complete in the class of topological semilattices if and only if every chain in  $E$  is finite. In [23] Gutik and Repovš established the closure of a linearly ordered topological semilattice in a topological semilattice. They proved the criterium of  $H$ -closedness of a linearly ordered topological semilattice in the class of topological semilattices and showed that every  $H$ -closed topological semilattice is absolutely  $H$ -closed in the class of topological semilattices. Also, such semilattices studied in [6, 14]. In [3] the structure of closures of the discrete semilattices  $(\mathbb{N}, \min)$  and  $(\mathbb{N}, \max)$  is described. Here the authors constructed an example of an  $H$ -closed topological semilattice in the class of topological semilattices which is not absolutely  $H$ -closed in the class of topological semilattices. The constructed example gives a negative answer on Question 17 from [32].

**Definition 1.1** ([19]). Let  $\mathfrak{STSG}_0$  be a class of semitopological semigroups. Let  $\lambda \geq 1$  be a cardinal and  $(S, \tau) \in \mathfrak{STSG}_0$ . Let  $\tau_B$  be a topology on  $B_\lambda^0(S)$  such that

- a)  $(B_\lambda^0(S), \tau_B) \in \mathfrak{STSG}_0$ ;
- b) the topological subspace  $(S_{\alpha, \alpha}, \tau_B|_{S_{\alpha, \alpha}})$  is naturally homeomorphic to  $(S, \tau)$  for some  $\alpha \in \lambda$ .

Then  $(B_\lambda^0(S), \tau_B)$  is called a *topological Brandt  $\lambda^0$ -extension of  $(S, \tau)$  in  $\mathfrak{STSG}_0$* .

In the paper [24] Gutik and Repovš established homomorphisms of the Brandt  $\lambda^0$ -extensions of monoids with zeros. They also described a category whose objects are ingredients in the constructions of the Brandt  $\lambda^0$ -extensions of monoids with zeros. Here they introduced finite, compact topological Brandt  $\lambda^0$ -extensions of topological semigroups and countably compact topological Brandt  $\lambda^0$ -extensions of topological inverse semigroups in the class of topological inverse semigroups, and established the structure of such extensions and non-trivial continuous homomorphisms between such topological Brandt  $\lambda^0$ -extensions of topological monoids with zero. There they also described a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt  $\lambda^0$ -extensions of topological monoids with zeros. These investigations were continued in [20, 21, 22], where established countably compact topological Brandt  $\lambda^0$ -extensions of topological monoids with zeros and pseudocompact topological Brandt  $\lambda^0$ -extensions of semitopological monoids with zeros their corresponding categories. In the papers [4, 15, 16, 19, 26] were studied  $H$ -closed and absolutely  $H$ -closed topological Brandt  $\lambda^0$ -extensions of topological semigroups in the class of topological semigroups.

In Section 2 we study the closure of a quasitopological group in a semitopological inverse semigroup with continuous inversion. In particular we show that a topological group  $G$  is  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion if and only if  $G$  is compact.

Section 3 is devoted to the closure of a semitopological semilattice in a semitopological inverse semigroup with continuous inversion. We show that a Hausdorff linearly ordered topological semilattice  $E$  is  $H$ -closed in the class of semitopological semilattices if and only if  $E$  is  $H$ -closed in the class of topological semilattices. Also, we construct an example of an  $H$ -closed semitopological semilattice in the class of semitopological semilattices which is not absolutely  $H$ -closed in the class of semitopological semilattices.

In Section 4 we show that a topological Brandt  $\lambda^0$ -extension of  $S$  is (absolutely)  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is  $S$ . Also, we study the preserving of (absolute)  $H$ -closedness in the class of semitopological inverse semigroups with continuous inversion by orthogonal sums.

## 2. ON THE CLOSURE OF A QUASITOPOLOGICAL GROUP IN A SEMITOPOLOGICAL INVERSE SEMIGROUP WITH CONTINUOUS INVERSION

**Proposition 2.1.** *Every left topological inverse semigroup with continuous inversion is semitopological semigroup.*

*Proof.* We write an arbitrary right translation  $\rho_a: S \rightarrow S: x \mapsto xa$  of a left topological inverse semigroup  $S$  with continuous inversion  $\mathbf{inv}: S \rightarrow S$  on three steps in the following way:

$$\rho_a(x) = xa = (a^{-1}x^{-1})^{-1} = (\mathbf{inv} \circ \lambda_{a^{-1}} \circ \mathbf{inv})(x).$$

This implies the continuity of right translations  $\rho_a$  on  $S$ . □

It is well known that the closure of an inverse subsemigroup of a topological inverse semigroup is again a topological inverse semigroup (see: [10, Proposition II.1]). The following proposition extends this result to semitopological inverse semigroups with continuous inversion.

**Proposition 2.2.** *The closure of an inverse subsemigroup  $T$  in a semitopological inverse semigroup  $S$  with continuous inversion is an inverse semigroup.*

*Proof.* By Proposition 1.8(ii) from [30, Chapter I, Proposition 1.8(ii)] the closure  $\text{cl}_S(T)$  of  $T$  in a semitopological semigroup  $S$  is a semitopological semigroup. Then the continuity of the inversion  $\mathbf{inv}: S \rightarrow S$  and Theorem 1.4.1 from [11] imply that  $\mathbf{inv}(\text{cl}_S(T)) \subseteq \text{cl}_S(\mathbf{inv}(T)) = \text{cl}_S(T)$  and hence we get that  $\mathbf{inv}(\text{cl}_S(T)) = \text{cl}_S(T)$ . This implies that  $\text{cl}_S(T)$  is an inverse subsemigroup of  $S$ . □

We observe that the statement of Proposition 2.2 is not true in the case of inverse topological semigroup. It is complete to consider the set  $\mathbb{R}^+ = [0, +\infty)$  of non-negative real numbers with usual topology and usual multiplication of real numbers. This implies that in Proposition 2.2 the condition that  $S$  has continuous inversion is essential.

In a compact topological semigroup the closure of a subgroup is a topological subgroup (see: [5, Vol. 1, Theorems 1.11 and 1.13]). Also, since for a topological inverse semigroup  $S$  the map  $f: S \rightarrow S: x \mapsto xx^{-1}$  is continuous, the maximal subgroup of  $S$  is closed, and hence the closure of a subgroup of a topological inverse semigroup is a subgroup. The previous observation implies that this is not true in the general case of topological semigroups. Also, the following example shows that the closure of a subgroup in a semitopological inverse semigroup with continuous inversion is not a subgroup.

**Example 2.3.** Let  $\mathbb{Z}$  be the discrete additive group of integers. We put  $\mathcal{A}(\mathbb{Z})$  is the one point Alexandroff compactification of the space  $\mathbb{Z}$  with the remainder  $\infty$ . We extend the semigroup operation from  $\mathbb{Z}$  onto  $\mathcal{A}(\mathbb{Z})$  in the following way:

$$n + \infty = \infty + n = \infty + \infty = \infty, \quad \text{for every } n \in \mathbb{Z}.$$

It is well known that  $\mathcal{A}(\mathbb{Z})$  with such defined operation is a semitopological inverse semigroup with continuous inversion and  $\mathbb{Z}$  is not a closed subgroup of  $\mathcal{A}(\mathbb{Z})$  [30].

A quasitopological group  $G$  is called *precompact* if for every open neighbourhood  $U$  of the neutral element of  $G$  there exists a finite subset  $F$  of  $G$  such that  $UF = G$  [2].

The following proposition gives examples quasitopological groups which are non-closed subgroups of some semitopological inverse semigroups with continuous inversion.

**Proposition 2.4.** *For every non-precompact regular quasitopological group  $(G, \tau)$  there exists a regular semitopological inverse semigroup with continuous inversion which contains  $(G, \tau)$  as a non-closed subgroup.*

*Proof.* Since the quasitopological group  $(G, \tau)$  is non-precompact there exists an open neighbourhood  $U$  of the neutral element  $e$  of the group  $G$  such that  $FU \neq G$  and  $UF \neq G$  for every finite subset  $F$  in  $G$ . Let  $\mathcal{B}_e$  be a base of the topology  $\tau$  at the neutral element  $e$  of  $(G, \tau)$ . Since the inversion is

continuous in  $(G, \tau)$ , without loss of generality we may assume that all elements of the family  $\mathcal{B}_e$  are symmetric, i.e.,  $V = V^{-1}$  for every  $V \in \mathcal{B}_e$ . We put

$$\mathcal{B}_U = \{V \in \mathcal{B}_e : \text{cl}_G(V) \subseteq U\}.$$

Since the quasitopological group  $(G, \tau)$  is not precompact we have that  $FV \neq G$  and  $VF \neq G$  for every  $V \in \mathcal{B}_U$  and for every finite subset  $F$  in  $G$ .

By  $G^0$  we denote the group  $G$  with a joined zero  $0$ . Now, we put

$$\mathcal{P}_0 = \{W_{g,V} = \{0\} \cup G \setminus \text{cl}_G(gV) : V \in \mathcal{B}_U, g \in G\} \cup \{W_{V,g} = \{0\} \cup G \setminus \text{cl}_G(Vg) : V \in \mathcal{B}_U, g \in G\}$$

and  $\tau \cup \mathcal{P}_0$  is a subbase of a topology  $\tau_0$  on  $G^0$ .

Since  $(G, \tau)$  a quasitopological group, it is sufficient to show that the semigroup operation on  $(G^0, \tau_0)$  is separately continuous in the following two cases:  $h \cdot 0 = 0$  and  $0 \cdot h = 0$ , for  $h \in G$ . Then for arbitrary subbase neighbourhoods  $W_{g_1, V_1}, \dots, W_{g_n, V_n}$  and  $W_{V_1, g_1}, \dots, W_{V_n, g_n}$  we have that

$$h \cdot (W_{g_1, V_1} \cap \dots \cap W_{g_n, V_n}) \subseteq W_{hg_1, V_1} \cap \dots \cap W_{hg_n, V_n}$$

and

$$(W_{V_1, g_1} \cap \dots \cap W_{V_n, g_n}) \cdot h \subseteq W_{V_1, g_1 h} \cap \dots \cap W_{V_n, g_n h}.$$

Also, since translations in the quasitopological group  $(G, \tau)$  are homeomorphisms, for every open subbase neighbourhood  $V \in \mathcal{B}_U$  of the neutral element of  $G$  and every  $g \in G$  we have that  $(W_{g,V})^{-1} \subseteq W_{V^{-1}, g^{-1}}$ . Therefore  $(G^0, \tau_0)$  is a quasitopological inverse semigroup with continuous inversion.

Now for every open subbase neighbourhoods  $V_1, V_2 \in \mathcal{B}_U$  of the neutral element of  $G$  such that  $\text{cl}_G(V_1) \subseteq V_2$  and every  $g \in G$  the following conditions holds:

$$\text{cl}_G(W_{g, V_2}) \subseteq W_{g, V_1} \quad \text{and} \quad \text{cl}_G(W_{V_2, g}) \subseteq W_{V_1, g}.$$

Hence we get that the topological space  $(G^0, \tau_0)$  is regular.  $\square$

**Theorem 2.5.** *A topological group  $G$  is  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion if and only if  $G$  is compact.*

*Proof.* The implication  $(\Leftarrow)$  is trivial.

$(\Rightarrow)$  Let a topological group  $G$  be  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion. Suppose to the contrary: the space  $G$  is not compact. Then  $G$  is  $H$ -closed in the class of topological groups and hence it is Raïkov complete. If  $G$  is precompact then by Theorem 3.7.15 of [2],  $G$  is compact. Hence the topological group  $G$  is not precompact. This contradicts Proposition 2.4. The obtained contradiction implies the statement of our theorem.  $\square$

Theorem 2.5 implies the following two corollaries:

**Corollary 2.6.** *A topological group  $G$  is absolutely  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion if and only if  $G$  is compact.*

**Corollary 2.7.** *A topological group  $G$  is  $H$ -closed in the class of semitopological semigroups if and only if  $G$  is compact.*

The following example shows that there exists a non-compact quasitopological group with adjoined zero which  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion.

**Example 2.8.** Let  $\mathbb{R}$  be the additive group of real numbers with usual topology. We put  $G$  is the direct square of  $\mathbb{R}$  with the product topology. It is well known that  $G$  is a topological group. Let  $G^0$  be the group  $G$  with the adjoined zero  $0$ . We define the topology  $\tau$  on  $G^0$  in the following way. For every non-zero element  $x$  of  $G^0$  the base of the topology  $\tau$  at  $x$  coincides with base of the product topology at  $x$  in  $G$ . For every  $(x_0, y_0) \in \mathbb{R}^2$  and every  $\varepsilon > 0$  we denote by

$$O_\varepsilon(x_0, y_0) = \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq \varepsilon \right\}$$



the usual closed  $\varepsilon$ -ball with the center at the point  $(x_0, y_0)$ . We denote

$$A(x_0, y_0) = \{(x_0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\} \cup \{(x, y_0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$$

and

$$U_\varepsilon(x_0, y_0) = G^0 \setminus (O_\varepsilon(x_0, y_0) \cup A(x_0, y_0)).$$

Now we put  $\mathcal{P}(0) = \{U_\varepsilon(x, y) : (x, y) \in \mathbb{R}^2, \varepsilon > 0\}$  and  $\mathcal{P}(0) \cup \mathcal{B}_G$  is a subbase of the topology  $\tau$  on  $G^0$ , where  $\mathcal{B}_G$  is a base of the topology of the topological group  $G$ . Simple verifications show that  $(G^0, \tau)$  is a Hausdorff semitopological inverse semigroup with continuous inversion and  $(G^0, \tau)$  is not a regular space.

Then for any finitely many points  $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$  and finitely many  $\varepsilon_1, \dots, \varepsilon_n > 0$  the following conditions hold:

- (a)  $O_{\varepsilon_1}(x_1, y_1) \cup \dots \cup O_{\varepsilon_n}(x_n, y_n)$  is a compact subset of the space  $(G^0, \tau)$ ;
- (b)  $\text{cl}_{G^0}(U_{\varepsilon_1}(x_1, y_1) \cap \dots \cap U_{\varepsilon_n}(x_n, y_n)) \cup O_{\varepsilon_1}(x_1, y_1) \cup \dots \cup O_{\varepsilon_n}(x_n, y_n) = G^0$ .

This implies that  $(G^0, \tau)$  is an  $H$ -closed topological space and hence the semigroup  $(G^0, \tau)$  is  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion.

### 3. ON THE CLOSURE OF A SEMILATTICE IN A SEMITOPOLOGICAL INVERSE SEMIGROUP WITH CONTINUOUS INVERSION

It is well known that the subset of idempotent  $E(S)$  of a topological semigroup  $S$  is a closed subset of  $S$  (see: [5, Vol. 1, Theorem 1.5]). We observe that for semitopological semigroups this statement does not hold [30]. Amassing, but the subset of all idempotent  $E(S)$  of a semitopological inverse semigroup  $S$  with continuous inversion is a closed subset of  $S$ .

**Proposition 3.1.** *The subset of idempotents  $E(S)$  of a semitopological inverse semigroup  $S$  with continuous inversion is a closed subset of  $S$ .*

*Proof.* First we observe that for any topological space  $X$  and any continuous map  $f: X \rightarrow X$  the set  $\text{Fix}(f)$  of fixed point of  $f$  is closed subset of  $X$  (see: [5, Vol. 1, Theorem 1.4] or [11, Theorem 1.5.4]). Since  $e^{-1} = e$  for every idempotent  $e \in S$ , the continuity of inversion implies that  $E(S) \subseteq \text{Fix}(\text{inv})$ . Let be  $x \in S$  such that  $x \in \text{Fix}(\text{inv})$ . Since  $S$  is an inverse semigroup we obtain that  $xx = xx^{-1} \in E(S)$  and hence  $\text{Fix}(\text{inv}) \subseteq E(S)$ . This completes the proof of the proposition.  $\square$

Proposition 3.1 implies the following

**Corollary 3.2.** *The closure of a subsemilattice in a semitopological inverse semigroup  $S$  with continuous inversion is a subsemilattice of  $S$ .*

Since the closure of a subsemilattice in a Hausdorff topological semigroup is again a topological semilattice, an (absolutely)  $H$ -closed topological semilattice in the class of topological semilattices is (absolutely)  $H$ -closed in the class of topological semigroups [16]. In [32] Stepp proved that an algebraic semilattice  $E$  is algebraically  $h$ -complete in the class of topological semilattices if and only if every chain in  $E$  is finite. The following example shows that for every infinite cardinal  $\lambda$  there exists an algebraically  $h$ -complete semilattice  $E(\lambda)$  in the class of topological semilattices of cardinality  $\lambda$  such that  $E(\lambda)$  with the discrete topology is not  $H$ -closed in the class of semitopological semigroups.

**Example 3.3.** Let  $\lambda$  be any infinite cardinal. We fix an arbitrary  $a_0 \in \lambda$  and define the semigroup operation on  $\lambda$  by the formula:

$$xy = \begin{cases} x, & \text{if } x = y; \\ a_0, & \text{if } x \neq y. \end{cases}$$

The cardinal  $\lambda$  with so defined semigroup operation we denote by  $E(\lambda)$ . It is obvious that  $E(\lambda)$  is a semilattice such that  $a_0$  is zero of  $E(\lambda)$  and any two distinct non-zero elements of  $E(\lambda)$  are incomparable

with respect to the natural partial order on  $E(\lambda)$ . Let be  $a \notin E(\lambda)$ . We extend the semigroup operation from  $E(\lambda)$  onto  $S = E(\lambda) \cup \{a\}$  in the following way:

$$aa = ax = xa = a_0, \quad \text{for any } x \in E(\lambda).$$

It is obvious that  $S$  with so defined operation is not a semilattice.

We define a topology  $\tau$  on  $S$  in the following way. Fix an arbitrary sequence of distinct points  $\{x_n: n \in \mathbb{N}\}$  from  $E(\lambda)$  and put  $U_n(a) = \{a\} \cup \{x_i: i \geq n\}$ . Put all elements of the set  $E(\lambda)$  are isolated points of the space  $(S, \tau)$  and the family  $\mathcal{B}(a) = \{U_n(a): n \in \mathbb{N}\}$  is a base of the topology  $\tau$  at the point  $a \in S$ . Simple verifications show that  $(S, \tau)$  is a metrizable 0-dimensional semitopological semigroup and  $E(\lambda)$  is a dense subsemilattice of  $(S, \tau)$ . Also, we observe that by Theorem 9 from [32] the semilattice  $E(\lambda)$  is algebraically  $h$ -complete in the class of topological semilattices.

**Remark 3.4.** We observe that for every infinite cardinal  $\lambda$  and every Hausdorff topology  $\tau$  on  $E(\lambda)$  such that  $(E(\lambda), \tau)$  is a semitopological semilattice we have that all non-zero idempotents of  $(E(\lambda), \tau)$  are isolated points and moreover  $(E(\lambda), \tau)$  is a topological semilattice. Also, a simple modification of the proof in the Example 3.3 shows that a semitopological semilattice  $(E(\lambda), \tau)$  is  $H$ -closed in the class of semitopological semigroups if and only if the space  $(E(\lambda), \tau)$  is compact.

Suppose that  $E$  is a Hausdorff semitopological semilattice. If  $L$  is a maximal chain in  $E$ , then by Proposition IV-1.13 of [12] we have that  $L = \bigcap_{e \in L} (\uparrow e \cup \downarrow e)$  is a closed subset of  $E$  and hence we proved the following proposition:

**Proposition 3.5.** *The closure of a linearly ordered subsemilattice of a Hausdorff semitopological semilattice  $E$  is a linearly ordered subsemilattice of  $E$ .*

It is well known that the natural partial order on a Hausdorff semitopological semilattice is semiclosed (see [12, Proposition IV-1.13]). Also, by Lemma 3 of [33] a semiclosed linear order is closed, and hence every linearly ordered set with a closed order admits the structure of a Hausdorff topological semilattice. This implies the following proposition:

**Proposition 3.6.** *Every linearly ordered Hausdorff semitopological semilattice is a topological semilattice.*

Propositions 3.5 and 3.6 imply

**Theorem 3.7.** *A Hausdorff linearly ordered topological semilattice  $E$  is  $H$ -closed in the class of semitopological semilattices if and only if  $E$  is  $H$ -closed in the class of topological semilattices.*

Theorem 3.7 and results obtained in the paper [23] imply Corollaries 3.8—3.12.

A linearly ordered semilattice  $E$  is called *complete* if every non-empty subset of  $S$  has inf and sup.

**Corollary 3.8.** *A linearly ordered semitopological semilattice  $E$  is  $H$ -closed in the class of semitopological semilattices if and only if the following conditions hold:*

- (i)  $E$  is complete;
- (ii)  $x = \sup A$  for  $A = \downarrow A \setminus \{x\}$  implies  $x \in \text{cl}_E A$ , whenever  $A \neq \emptyset$ ; and
- (iii)  $x = \inf B$  for  $B = \uparrow B \setminus \{x\}$  implies  $x \in \text{cl}_E B$ , whenever  $B \neq \emptyset$

**Corollary 3.9.** *Every linearly ordered  $H$ -closed semitopological semilattice in the class of semitopological semilattices is absolutely  $H$ -closed in the class of semitopological semilattices.*

**Corollary 3.10.** *Every linearly ordered  $H$ -closed semitopological semilattice in the class of semitopological semilattices contains maximal and minimal idempotents.*

**Corollary 3.11.** *Let  $E$  be a linearly ordered  $H$ -closed semitopological semilattice in the class of semitopological semilattices and  $e \in E$ . Then  $\uparrow e$  and  $\downarrow e$  are (absolutely)  $H$ -closed topological semilattices in the class of semitopological semilattices.*

**Corollary 3.12.** *Every linearly ordered semitopological semilattice is a dense subsemilattice of an  $H$ -closed semitopological semilattice in the class of semitopological semilattices.*

**Remark 3.13.** Theorem 3.7, Example 7 and Proposition 8 from [23] imply that there exists a countable linearly ordered  $\sigma$ -compact 0-dimensional scattered locally compact metrizable topological semilattice which does not embeds into any compact Hausdorff semitopological semilattice.

At the finish of this section we construct an  $H$ -closed semitopological semilattice in the class of semitopological semilattices which is not absolutely  $H$ -closed in the class of semitopological semilattices.

A filter  $\mathcal{F}$  on a set  $X$  is called *free* if  $\bigcap \mathcal{F} = \emptyset$ .

**Example 3.14** ([3]). Let  $\mathbb{N}$  denote the set of positive integers. For each free filter  $\mathcal{F}$  on  $\mathbb{N}$  consider the topological space  $\mathbb{N}_{\mathcal{F}} = \mathbb{N} \cup \{\mathcal{F}\}$  in which all points  $x \in \mathbb{N}$  are isolated while the sets  $F \cup \{\mathcal{F}\}$ ,  $F \in \mathcal{F}$ , form a neighbourhood base at the unique non-isolated point  $\mathcal{F}$ .

The semilattice operation  $\min$  of  $\mathbb{N}$  extends to a continuous semilattice operation  $\min$  on  $\mathbb{N}_{\mathcal{F}}$  such that  $\min\{n, \mathcal{F}\} = \min\{\mathcal{F}, n\} = n$  and  $\min\{\mathcal{F}, \mathcal{F}\} = \mathcal{F}$  for all  $n \in \mathbb{N}$ . By  $\mathbb{N}_{\mathcal{F}, \min}$  we shall denote the topological space  $\mathbb{N}_{\mathcal{F}}$  with the semilattice operation  $\min$ . Simple verifications show that  $\mathbb{N}_{\mathcal{F}, \min}$  is a topological semilattice. Then by Theorem 2(i) of [3] the topological semilattice  $\mathbb{N}_{\mathcal{F}, \min}$  is  $H$ -closed in the class of topological semilattices and hence by Theorem 3.7 it is  $H$ -closed in the class of semitopological semilattices.

Later by  $E_2 = \{0, 1\}$  we denote the discrete topological semilattice with the semilattice operation  $\min$ .

**Theorem 3.15.** *Let  $\mathcal{F}$  be a free filter on  $\mathbb{N}$  and  $F \in \mathcal{F}$  be a set with infinite complement  $\mathbb{N} \setminus F$ . Then the closed subsemilattice  $E = (\mathbb{N}_{\mathcal{F}, \min} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{1\})$  of the direct product  $\mathbb{N}_{\mathcal{F}, \min} \times E_2$  is  $H$ -closed not absolutely  $H$ -closed in the class of semitopological semilattices.*

*Proof.* The definition of the topological semilattice  $\mathbb{N}_{\mathcal{F}, \min} \times E_2$  implies that  $E$  is a closed subsemilattice of  $\mathbb{N}_{\mathcal{F}, \min} \times E_2$ .

Suppose the contrary: the topological semilattice  $E$  is not  $H$ -closed in the class of semitopological semilattices. Since the closure of a subsemilattice in a semitopological semilattice is a semilattice (see [30, Chapter I, Proposition 1.8(ii)]) we conclude that there exists a semitopological semilattice  $S$  which contains  $E$  as a dense subsemilattice and  $S \setminus E \neq \emptyset$ . We fix an arbitrary  $a \in S \setminus E$ . Then for every open neighbourhood  $U(a)$  of the point  $a$  in  $S$  we have that the set  $U(a) \cap E$  is infinite. By Theorem 2(i) of [3] and Theorem 3.7, the subspace  $\mathbb{N}_{\mathcal{F}, \min} \times \{0\}$  of  $E$  with the induced semilattice operation from  $E$  is an  $H$ -closed in the class of semitopological semilattices. Therefore there exists an open neighbourhood  $U(a)$  of the point  $a$  in  $S$  such that  $U(a) \cap E \subseteq (\mathbb{N} \setminus F) \times \{1\}$  and hence the set  $U(a) \cap ((\mathbb{N} \setminus F) \times \{1\})$  is infinite.

Since the subset  $\mathbb{N}_{\mathcal{F}, \min} \times \{0\}$  is an ideal of  $E$ , the  $H$ -closedness of  $\mathbb{N}_{\mathcal{F}, \min} \times \{0\}$  in the class of semitopological semilattices implies that  $\mathbb{N}_{\mathcal{F}, \min} \times \{0\}$  is a closed ideal in  $S$  and hence we have that  $x \cdot a \in \mathbb{N}_{\mathcal{F}, \min} \times \{0\}$  for every  $x \in \mathbb{N}_{\mathcal{F}, \min} \times \{0\}$ . Since for every open neighbourhood  $U(a)$  of the point  $a$  in  $S$  the set  $U(a) \cap ((\mathbb{N} \setminus F) \times \{1\})$  is infinite the semilattice operation in  $E$  implies that for every  $x \in (\mathbb{N}_{\mathcal{F}, \min} \times \{0\}) \setminus \{(\mathcal{F}, 0)\}$  the set  $x \cdot U(a)$  is infinite and hence we have that  $x \cdot a \notin \mathbb{N} \times \{0\} = (\mathbb{N}_{\mathcal{F}, \min} \times \{0\}) \setminus \{(\mathcal{F}, 0)\}$ . Therefore we obtain that  $x \cdot a = (\mathcal{F}, 0)$ . Now, since in  $\mathbb{N}_{\mathcal{F}, \min}$  the sets  $F \cup \{\mathcal{F}\}$ ,  $F \in \mathcal{F}$ , form a neighbourhood base at the unique non-isolated point  $\mathcal{F}$ , we conclude that  $x \cdot U(a) \not\subseteq (F \cup \{\mathcal{F}\}) \times \{0\}$ , which contradicts the separate continuity of the semilattice operation on  $S$ . Hence we get that  $S \setminus E = \emptyset$ . This implies that the topological semilattice  $E$  is  $H$ -closed in the class of semitopological semilattices.

Now, by Theorem 3 of [3] the topological semilattice  $E$  is not absolutely  $H$ -closed in the class of topological semilattices, and hence  $E$  is not absolutely  $H$ -closed in the class of semitopological semilattices.  $\square$



**Remark 3.16.** Corollary 3.2 implies that the topological semilattice  $E$  determined in Theorem 3.15 is an example a topological inverse semigroup which is  $H$ -closed but is not absolutely  $H$ -closed in the class of semitopological semigroups with continuous inversion.

**Remark 3.17.** Proposition 3.6 and Theorem 3.7 imply that Theorem 2 of [3] describes all  $H$ -closed semilattices in the class of semitopological semilattices which contain the discrete semilattice  $(\mathbb{N}, \min)$  or the discrete semilattice  $(\mathbb{N}, \max)$  as a dense subsemilattice.

#### 4. ON THE CLOSURE OF TOPOLOGICAL BRANDT $\lambda$ -EXTENSIONS IN A SEMITOPOLOGICAL INVERSE SEMIGROUP WITH CONTINUOUS INVERSION

In this section we study the preserving of  $H$ -closedness and absolute  $H$ -closedness by topological Brandt  $\lambda^0$ -extensions and orthogonal sums of semitopological semigroups.

**Theorem 4.1.** *Let  $S$  be a Hausdorff semitopological inverse monoid with zero and continuous inversion. Then the following conditions are equivalent:*

- (i)  *$S$  is absolutely  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion;*
- (ii) *there exists a cardinal  $\lambda \geq 2$  such that every topological Brandt  $\lambda^0$ -extension of  $S$  is absolutely  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion;*
- (iii) *for each cardinal  $\lambda \geq 2$  every topological Brandt  $\lambda^0$ -extension of  $S$  is absolutely  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion.*

*Proof.* (i)  $\Rightarrow$  (iii). Suppose that the semigroup  $S$  is absolutely  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion. We fix an arbitrary cardinal  $\lambda \geq 2$ . Let  $B_\lambda^0(S)$  be a topological Brandt  $\lambda^0$ -extension of  $S$  in the class of semitopological inverse semigroups with continuous inversion,  $T$  be a semitopological inverse semigroup with continuous inversion and  $h: B_\lambda^0(S) \rightarrow T$  be a continuous homomorphism.

First we observe that by Proposition 2.3 of [24], either  $h$  is an annihilating homomorphism or the image  $(B_\lambda^0(S))h$  is isomorphic to the Brandt  $\lambda^0$ -extension  $B_\lambda^0((S_{\alpha,\alpha})h)$  of the semigroup  $(S_{\alpha,\alpha})h$  for some  $\alpha \in \lambda$ . If  $h$  is an annihilating homomorphism then  $(S_{\alpha,\alpha})h$  is a singleton, and therefore we have that  $(S_{\alpha,\alpha})h$  is a closed subset of  $T$ . Hence, later we assume that  $h$  is a non-annihilating homomorphism.

Next we show that for any  $\gamma, \delta \in \lambda$  the set  $(S_{\gamma,\delta})h$  is closed in the space  $T$ . By Definition 1.1 there exists  $\alpha \in \lambda$  such that  $(S_{\alpha,\alpha})h$  is a closed subset of  $T$ . We define the maps  $\varphi_h, \psi_h: T \rightarrow T$  by the formulae  $(x)\varphi_h = (\alpha, 1_S, \gamma)h \cdot (x)h \cdot (\delta, 1_S, \alpha)h$  and  $(x)\psi_h = (\gamma, 1_S, \alpha)h \cdot (x)h \cdot (\alpha, 1_S, \delta)h$ . Then the maps  $\varphi_h$  and  $\psi_h$  are continuous because left and right translations in  $T$  and homomorphism  $h: B_\lambda^0(S) \rightarrow T$  are continuous maps. Thus, the full preimage  $A = ((S_{\alpha,\alpha})h)\varphi_h^{-1}$  is a closed subset of  $T$ . Then the restriction map  $(\varphi_h \circ \psi_h)|_A: A \rightarrow (S_{\gamma,\delta})h$  is a retraction, and therefore the set  $(S_{\gamma,\delta})h$  is a retract of  $A$ . This implies that  $(S_{\gamma,\delta})h$  is a closed subset of  $T$ .

Suppose to the contrary that  $(B_\lambda^0(S))h$  is not a closed subsemigroup of  $T$ . By Lemma II.1.10 of [27],  $(B_\lambda^0(S))h$  is an inverse subsemigroup of  $T$ . Since by Proposition 2.2 the closure of an inverse subsemigroup  $(B_\lambda^0(S))h$  in a semitopological inverse semigroup  $T$  with continuous inversion is an inverse semigroup, without loss of generality we may assume that  $(B_\lambda^0(S))h$  is a dense proper inverse subsemigroup of  $T$ .

We fix an arbitrary  $x \in \text{cl}_T((B_\lambda^0(S))h) \setminus (B_\lambda^0(S))h$ . Then only one of the following cases holds:

- a)  $x$  is an idempotent of the semigroup  $T$ ;
- b)  $x$  is a non-idempotent element of  $T$ .

Suppose that case a) holds. By the previous part of the proof we have that every open neighbourhood  $U(x)$  of the point  $x$  in the topological space  $T$  intersects infinitely sets of the form  $(S_{\alpha,\beta})h$ ,  $\alpha, \beta \in \lambda$ . By Proposition 2.3 of [24],  $(B_\lambda^0(S))h$  is isomorphic to the Brandt  $\lambda^0$ -extension  $B_\lambda^0((S_{\alpha,\alpha})h)$  of the semigroup  $(S_{\alpha,\alpha})h$  for some  $\alpha \in \lambda$ , and since  $(B_\lambda^0(S))h$  is a dense subsemigroup of semitopological semigroup  $T$ , the zero 0 of the semigroup  $(B_\lambda^0(S))h$  is zero of  $T$  (see [20, Lemma 23]). Then the semigroup operation of  $B_\lambda^0((S_{\alpha,\alpha})h)$  implies that either  $0 \in (\alpha, e, \alpha)h \cdot U(x)$  or  $0 \in U(x) \cdot (\alpha, e, \alpha)h$  for every non-zero idempotent

$(\alpha, e, \alpha)$  of  $B_\lambda^0(S)$ ,  $e \in E(S)$ ,  $\alpha \in \lambda$ . Now by the Hausdorffness of the space  $T$  and the separate continuity of the semigroup operation of  $T$  we have that either  $(\alpha, e, \alpha)h \cdot x = 0$  or  $x \cdot (\alpha, e, \alpha)h = 0$  for every non-zero idempotent  $(\alpha, e, \alpha)$  of  $B_\lambda^0(S)$ ,  $e \in E(S)$ ,  $\alpha \in \lambda$ . Since in an inverse semigroup any two idempotents commute we conclude that  $(\alpha, e, \alpha)h \cdot x = x \cdot (\alpha, e, \alpha)h = 0$  for every non-zero idempotent  $(\alpha, e, \alpha)$  of the semigroup  $B_\lambda^0(S)$ .

We fix an arbitrary non-zero element  $(\alpha, s, \beta)h$  of the semigroup  $(B_\lambda^0(S))h$ , where  $\alpha, \beta \in \lambda$  and  $s \in S^*$ . Then by the previous part of the proof we obtain that

$$x \cdot (\alpha, s, \beta)h = x \cdot (\alpha, ss^{-1}s, \beta)h = x \cdot ((\alpha, ss^{-1}, \alpha)(\alpha, s, \beta))h = x \cdot (\alpha, ss^{-1}, \alpha)h \cdot (\alpha, s, \beta)h = 0 \cdot (\alpha, s, \beta)h = 0$$

and

$$(\alpha, s, \beta)h \cdot x = (\alpha, ss^{-1}s, \beta)h \cdot x = ((\alpha, s, \beta)(\beta, s^{-1}s, \beta))h \cdot x = (\alpha, s, \beta)h \cdot (\beta, s^{-1}s, \beta)h \cdot x = (\alpha, s, \beta)h \cdot 0 = 0.$$

This implies that for every open neighbourhood  $U(x)$  of the point  $x$  in the space  $T$  we have that  $0 \in x \cdot U(x)$  and  $0 \in U(x) \cdot x$ . Then by Hausdorffness of the space  $T$  and the separate continuity of the semigroup operation in  $T$  we get that  $x \cdot x = 0$ , and hence  $x = 0$ . This implies that  $E(T) = E((B_\lambda^0(S))h)$ .

Suppose that case *b*) holds. If  $xx^{-1} = 0$ , then  $x = xx^{-1}x = 0 \cdot x = 0$ , and similarly if  $x^{-1}x = 0$ , then  $x = xx^{-1}x = x \cdot 0 = 0$ . This implies that  $xx^{-1}, x^{-1}x \in E((B_\lambda^0(S))h) \setminus \{0\}$ .

Then by Lemma I.7.10 of [27] there exist idempotents  $(\alpha, e, \alpha), (\beta, f, \beta) \in B_\lambda^0(S)$  such that  $xx^{-1} = (\alpha, e, \alpha)h$  and  $x^{-1}x = (\beta, f, \beta)h$ , where  $e, f \in (E(S))^*$  and  $\alpha, \beta \in \lambda$ . Then we have that  $x \cdot (\beta, f, \beta)h = (\alpha, e, \alpha)h \cdot x = x$ . Since  $x \in \text{cl}_T((B_\lambda^0(S))h) \setminus (B_\lambda^0(S))h$ , every open neighbourhood  $U(x)$  of the point  $x$  in the space  $T$  intersects infinitely many sets  $(S_{\gamma, \delta})h$ ,  $\gamma, \delta \in \lambda$ , and hence we obtain that either  $U(x) \cdot (\beta, f, \beta)h \ni 0$  or  $(\alpha, e, \alpha)h \cdot U(x) \ni 0$ . Then the Hausdorffness of the space  $T$  and the separate continuity of the semigroup operation on  $T$  imply that  $x \cdot (\beta, f, \beta)h = 0$  or  $(\alpha, e, \alpha)h \cdot x = 0$ . If  $x \cdot (\beta, f, \beta)h = 0$  then  $x = x \cdot xx^{-1} = x \cdot (\alpha, e, \alpha)h = 0$  and if  $(\alpha, e, \alpha)h \cdot x = 0$  then  $x = xx^{-1}x = (\alpha, e, \alpha)h \cdot x = 0$ . All these two cases imply that  $x = 0$ , and hence we get that  $T = (B_\lambda^0(S))h$ , which completes the proof of our theorem.

The implication  $(iii) \Rightarrow (ii)$  is trivial.

$(ii) \Rightarrow (i)$ . Suppose to the contrary: there exists semigroup  $S$  such that  $S$  is not absolutely  $H$ -closed semigroup  $S$  in the class of semitopological inverse semigroups with continuous inversion and condition  $(ii)$  holds for  $S$ . Then there exists a semitopological inverse semigroup  $T$  with continuous inversion and continuous homomorphism  $h: S \rightarrow T$  such that  $(S)h$  is non-closed subset of  $T$ . Now, by Proposition 2.2 without loss of generality we may assume that  $(S)h$  is a proper dense inverse subsemigroup of  $T$ .

Next, for the cardinal  $\lambda$  we define topologies  $\tau_T^B$  and  $\tau_S^B$  on Brandt  $\lambda^0$ -extensions  $B_\lambda^0(T)$  and  $B_\lambda^0(S)$ , respectively, in the following way. We put

$$\mathcal{B}_{(\alpha, t, \beta)}^T = \{(U(t))_{\alpha, \beta}: 0 \notin U(t) \in \mathcal{B}_T(t)\} \quad \text{and} \quad \mathcal{B}_{(\alpha, s, \beta)}^S = \{(U(s))_{\alpha, \beta}: 0 \notin U(s) \in \mathcal{B}_S(s)\}$$

are bases of topologies  $\tau_T^B$  and  $\tau_S^B$  at non-zero elements  $(\alpha, t, \beta) \in B_\lambda^0(T)$  and  $(\alpha, s, \beta) \in B_\lambda^0(S)$ , respectively,  $\alpha, \beta \in \lambda$ , where  $\mathcal{B}_T(t)$  and  $\mathcal{B}_S(s)$  are bases of topologies of spaces  $T$  and  $S$  at non-zero elements  $t \in T$  and  $s \in S$ , respectively. Also, if  $\mathcal{B}_T(0_T)$  and  $\mathcal{B}_S(0_S)$  are bases at zeros  $0_T \in T$  and  $0_S \in S$  then we define

$$\mathcal{B}_0^T = \left\{ \{0\} \cup \bigcup_{\alpha, \beta \in \lambda} (U(0_T))_{\alpha, \beta}^*: U(0_T) \in \mathcal{B}_T(0_T) \right\} \quad \text{and} \quad \mathcal{B}_0^S = \left\{ \{0\} \cup \bigcup_{\alpha, \beta \in \lambda} (U(0_S))_{\alpha, \beta}^*: U(0_S) \in \mathcal{B}_S(0_S) \right\}$$

to be the bases of topologies  $\tau_T^B$  and  $\tau_S^B$  at zeros  $0 \in B_\lambda^0(T)$  and  $0 \in B_\lambda^0(S)$ , respectively.

Simple verifications show that if  $T$  and  $S$  are semitopological inverse semigroups with continuous inversion, then so are  $(B_\lambda^0(T), \tau_T^B)$  and  $(B_\lambda^0(S), \tau_S^B)$ . Also the continuity of homomorphism  $h: S \rightarrow T$  implies that the map  $h_B: B_\lambda^0(S) \rightarrow B_\lambda^0(T)$  defined by the formulae

$$(\alpha, s, \beta)h_B = \begin{cases} (\alpha, (s)h, \beta), & \text{if } (s)h \neq 0_T; \\ 0, & \text{otherwise,} \end{cases}$$

$s \in S^*$ ,  $\alpha, \beta \in \lambda$ , and  $(0)h_B = 0$  is continuous. Also, by Theorem 3.10 of [24] so defines map  $h_B: B_\lambda^0(S) \rightarrow B_\lambda^0(T)$  is a homomorphism. The definition of the topology  $\tau_T^B$  on  $B_\lambda^0(T)$  implies that the

homomorphic image  $(B_\lambda^0(S))h_B$  is a dense proper subsemigroup of the semitopological inverse semigroup  $(B_\lambda^0(T), \tau_T^B)$  with continuous inversion, which contradicts to statement (ii). The obtained contradiction implies the requested implication.  $\square$

Now, if we put  $h$  is a topological isomorphic embedding of semitopological semigroups with continuous inversions in the proof of Theorem 4.1, then we get the proof of the following theorem:

**Theorem 4.2.** *Let  $S$  be a Hausdorff semitopological inverse monoid with zero and continuous inversion. Then the following conditions are equivalent:*

- (i)  *$S$  is  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion;*
- (ii) *there exists a cardinal  $\lambda \geq 2$  such that every topological Brandt  $\lambda^0$ -extension of  $S$  is  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion;*
- (iii) *for each cardinal  $\lambda \geq 2$  every topological Brandt  $\lambda^0$ -extension of  $S$  is  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion.*

Theorem 4.1 implies Corollary 4.3 which generalizes Corollary 20 from [17].

**Corollary 4.3.** *For any cardinal  $\lambda \geq 2$  the semigroup of  $\lambda \times \lambda$ -units  $B_\lambda$  is algebraically  $h$ -complete in the class of semitopological inverse semigroups with continuous inversion.*

Also, Theorems 4.1 and 4.2 imply the following corollary:

**Corollary 4.4.** *For an inverse monoid  $S$  with zero the following conditions are equivalent:*

- (i)  *$S$  is algebraically complete (algebraically  $h$ -complete) in the class of semitopological inverse semigroups with continuous inversion;*
- (ii) *there exists a cardinal  $\lambda \geq 2$  such the Brandt  $\lambda^0$ -extension of  $S$  is algebraically complete (algebraically  $h$ -complete) in the class of semitopological inverse semigroups with continuous inversion;*
- (iii) *for each cardinal  $\lambda \geq 2$  the Brandt  $\lambda^0$ -extension of  $S$  is algebraically complete (algebraically  $h$ -complete) in the class of semitopological inverse semigroups with continuous inversion.*

Theorems 4.5, 4.6 and 4.7 give a method of the construction of absolutely  $H$ -closed and  $H$ -closed semigroups in the class of semitopological inverse semigroups with continuous inversion.

**Theorem 4.5.** *Let  $S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$  be a semitopological inverse semigroup with continuous inversion such that*

- (i)  *$S_\alpha$  is an absolutely  $H$ -closed semigroup in the class of semitopological inverse semigroups with continuous inversion for any  $\alpha \in \mathcal{A}$ ; and*
- (ii) *there exists an ideal  $T$  of  $S$  which is absolutely  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion such that  $S_\alpha \cdot S_\beta \subseteq T$  for all  $\alpha \neq \beta$ ,  $\alpha, \beta \in \mathcal{A}$ .*

*Then  $S$  is an absolutely  $H$ -closed semigroup in the class of semitopological inverse semigroups with continuous inversion.*

*Proof.* Suppose to the contrary that there exists a semitopological inverse semigroup  $K$  with continuous inversion and continuous homomorphism  $h: S \rightarrow K$  such that the image  $(S)h$  is not a closed subsemigroup of  $K$ . By Lemma II.1.10 of [27],  $(S)h$  is an inverse subsemigroup of  $K$ . Since by Proposition 2.2 the closure  $\text{cl}_K((S)h)$  of an inverse subsemigroup  $(S)h$  in a semitopological inverse semigroup  $K$  with continuous inversion is an inverse semigroup, without loss of generality we may assume that  $(S)h$  is a dense proper inverse subsemigroup of  $K$ .

We observe that the assumption of the theorem states that  $T$  is an ideal of  $S$ . This implies that  $(T)h$  is an ideal in  $(S)h$ . Then by Proposition I.1.8(iii) of [30] the closure of an ideal of a semitopological semigroup is again an ideal, and hence we get that  $(T)h$  is a closed ideal of the semigroup  $K$ .

We fix an arbitrary  $x \in K \setminus (S)h$ . Then only one of the following cases holds:

- a)  $x$  is an idempotent of the semigroup  $K$ ;
- b)  $x$  is a non-idempotent element of  $K$ .

First we show that  $x \cdot y, y \cdot x \in (T)h$  for every  $y \in (S)h$ . We fix an arbitrary open neighbourhood  $U(x)$  of the point  $x$  in the space  $K$ . Since  $U(x)$  intersects infinitely many subsemigroups of  $K$  from the family  $\{(S_\alpha)h : \alpha \in \mathcal{A}\}$  we conclude that  $U(x) \cdot y \cap (T)h \neq \emptyset$  and  $y \cdot U(x) \cap (T)h \neq \emptyset$  for every  $y \in (S)h$ . Then the separate continuity of the semigroup operation in  $K$  implies that any open neighbourhoods  $W(x \cdot y)$  and  $W(y \cdot x)$  of the points  $x \cdot y$  and  $y \cdot x$  in  $K$ , respectively, intersect the ideal  $(T)h$ . This implies that  $x \cdot y, y \cdot x \in \text{cl}_K((T)h)$ . Since the ideal  $(T)h$  is closed in  $K$  we conclude that  $x \cdot y, y \cdot x \in (T)h$ .

Suppose that case a) holds. Then there exists an open neighbourhood  $U(x)$  of the point  $x$  in the space  $K$  such that  $U(x) \cap (T)h = \emptyset$  and the neighbourhood  $U(x)$  intersects infinitely many semigroups from the family  $\{(S_\alpha)h : \alpha \in \mathcal{A}\}$ . By the separate continuity of the semigroup operation in  $K$  we have that for every open neighbourhood  $U(x)$  of the point  $x$  in  $K$  such that  $U(x) \cap (T)h = \emptyset$  there exists an open neighbourhood  $V(x)$  of  $x$  in  $K$  such that  $x \cdot V(x) \subseteq U(x)$  and  $V(x) \cdot x \subseteq U(x)$ . Now, the previous part of proof implies that  $x \cdot V(x) \cap (T)h \neq \emptyset$  and  $V(x) \cdot x \cap (T)h \neq \emptyset$ , which contradict the assumption  $U(x) \cap (T)h = \emptyset$ . The obtained contradiction implies that  $E((S)h) = E(K)$ .

Suppose that case b) holds. Then there exist idempotents  $e$  and  $f$  in  $(S)h$  such that  $xx^{-1} = e$  and  $x^{-1}x = f$ . We observe that  $e, f \notin (T)h$ . Indeed, if  $e \in (T)h$  or  $f \in (T)h$ , then we have that

$$x = xx^{-1}x = ex \in (T)h \quad \text{and} \quad x = xx^{-1}x = xf \in (T)h,$$

because  $(T)h$  is an ideal of the semigroup  $K$ . Since  $x \in \text{cl}_K((S)h)$ , every open neighbourhood of the point  $x$  in  $K$  intersects infinitely many semigroups from the family  $\{(S_\alpha)h : \alpha \in \mathcal{A}\}$ , and hence we get that

$$(U(x) \cdot f) \cap (T)h \neq \emptyset \quad \text{and} \quad (e \cdot U(x)) \cap (T)h \neq \emptyset.$$

Then the Hausdorffness of  $K$  and the separate continuity of the semigroup operation in  $K$  imply that  $x = xx^{-1}x = x \cdot f = e \cdot x \in (T)h$ . This contradicts the assumption that  $x \notin (T)h$ . The obtained contradiction implies the statement of our theorem.  $\square$

The proof of Theorem 4.6 is similar to the proof of Theorem 4.5.

**Theorem 4.6.** *Let  $S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$  be a semitopological inverse semigroup with continuous inversion such that*

- (i)  $S_\alpha$  is an  $H$ -closed semigroup in the class of semitopological inverse semigroups with continuous inversion for any  $\alpha \in \mathcal{A}$ ; and
- (ii) there exists an ideal  $T$  of  $S$  which is  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion such that  $S_\alpha \cdot S_\beta \subseteq T$  for all  $\alpha \neq \beta$ ,  $\alpha, \beta \in \mathcal{A}$ .

*Then  $S$  is an  $H$ -closed semigroup in the class of semitopological inverse semigroups with continuous inversion.*

**Theorem 4.7.** *Let a semitopological semigroup  $S$  with continuous inversion be the orthogonal sum of a family  $\{S_\alpha : \alpha \in \mathcal{J}\}$  of semitopological inverse semigroups with zeros. Then  $S$  is an (absolutely)  $H$ -closed semigroup in the class of semitopological inverse semigroups with continuous inversion if and only if so is any element of the family  $\{S_\alpha : \alpha \in \mathcal{J}\}$ .*

*Proof.* First we observe that if  $S$  is a semitopological semigroup with continuous inversion then so is every semigroup from the family  $\{S_\alpha : \alpha \in \mathcal{J}\}$ .

The implication  $(\Leftarrow)$  follows from Theorems 4.5 and 4.6.

First we shall prove the implication  $(\Rightarrow)$  in the case of absolute  $H$ -closedness.

Suppose to the contrary that there exists an absolute  $H$ -closed semigroup  $S$  in the class of semitopological inverse semigroups with continuous inversion which is an orthogonal sum of a family  $\{S_\alpha : \alpha \in \mathcal{J}\}$  of semitopological inverse semigroups and there exists a semigroup  $S_{\alpha_0}$  in this family such that  $S_{\alpha_0}$  is not absolute  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion. Then there exists a semitopological inverse semigroup  $K$  with continuous inversion and continuous homomorphism  $h : S_{\alpha_0} \rightarrow K$  such that the image  $(S_{\alpha_0})h$  is not a closed subsemigroup of  $K$ . By Lemma II.1.10 of [27],  $(S_{\alpha_0})h$  is an inverse subsemigroup of  $K$ . Since by Proposition 2.2 the closure  $\text{cl}_K((S_{\alpha_0})h)$  of an inverse subsemigroup  $(S_{\alpha_0})h$  in a semitopological inverse semigroup  $K$  with continuous inversion is



an inverse semigroup, without loss of generality we may assume that  $(S_{\alpha_0})h$  is a dense proper inverse subsemigroup of  $K$ . Also, the semigroup  $K$  has zero because  $(S_{\alpha_0})h$  contains zero.

We define a map  $f: S \rightarrow K$  by the formula

$$(x)f = \begin{cases} 0_K, & \text{if } x \in S \setminus S_{\alpha_0}^*; \\ (x)h, & \text{if } x \in S_{\alpha_0}^*, \end{cases}$$

where  $0_K$  is zero of the semigroup  $K$ . Simple verifications show that so defined map  $f$  is a continuous homomorphism, but the image  $(S)f = (S_{\alpha_0})h$  is a dense proper subsemigroup of  $K$ . This contradicts the assumption that the semigroup  $S$  is absolutely  $H$ -closed semigroup in the class of semitopological inverse semigroups with continuous inversion.

Now, we suppose that there exists an  $H$ -closed semigroup  $S$  in the class of semitopological inverse semigroups with continuous inversion which is an orthogonal sum of a family  $\{S_\alpha: \alpha \in \mathcal{J}\}$  of semitopological inverse semigroups and there exists a semigroup  $S_{\alpha_0}$  in this family such that  $S_{\alpha_0}$  is not  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion. Then there exists a semitopological inverse semigroup  $K$  with continuous inversion such that  $S_{\alpha_0}$  is not a closed subsemigroup of  $K$ . Since by Proposition 2.2 the closure  $\text{cl}_K(S_{\alpha_0})$  of an inverse subsemigroup  $S_{\alpha_0}$  in a semitopological inverse semigroup  $K$  with continuous inversion is an inverse semigroup, without loss of generality we may assume that  $S_{\alpha_0}$  is a dense proper inverse subsemigroup of  $K$ .

Next, we put  $S'$  be the orthogonal sum of the family  $\{S_\alpha: \alpha \in \mathcal{J} \setminus \{\alpha_0\}\}$  and the semigroup  $K$ . We determine a topology  $\tau$  on  $S'$  in the following way.

First we observe if the orthogonal sum  $T = \sum_{i \in \mathcal{J}} T_j$  is an inverse Hausdorff semitopological semigroup, then for every non-zero element  $t \in T_j \subset T$  there exists an open neighbourhood  $U(t)$  of  $t$  in  $T$  such that  $U(t) \subseteq T_j^*$ . Indeed, for every open neighbourhood  $W(t) \not\ni 0$  of  $t$  in  $T$  there exists an open neighbourhood  $U(t)$  of  $t$  in  $T$  such that  $tt^{-1} \cdot U(t) \subseteq W(t)$ . The neighbourhood  $U(t)$  is requested.

We put that the bases of topologies at any point  $s$  of  $S \setminus S_{\alpha_0}$  and of  $S' \setminus K$  coincide in  $S$  and in  $S'$ , respectively. Also the bases at any point  $s$  of subspace  $K^* \subseteq S'$  coincide with the base at the point  $s$  of  $K^*$ . The following family determines the base of the topology  $\tau$  at zero of the semigroup  $S'$ :

$$\begin{aligned} \mathcal{B}_0 = \{ & U \subseteq S': \text{there exist an element } V \text{ of the base at zero of the topology of } S \\ & \text{and an element } W \text{ of the base at zero of the topology of } K \text{ such that} \\ & U \cap S' \setminus K = V \cap S \setminus S_{\alpha_0}, U \cap K = W \text{ and } U \cap S_{\alpha_0} = W \cap S_{\alpha_0} \}. \end{aligned}$$

Simple verifications show that  $(S', \tau)$  is a Hausdorff semitopological inverse semigroup with continuous inversion and moreover  $S$  is a dense proper inverse subsemigroup of  $(S', \tau)$ , which contradicts the assumption of our theorem. The obtained contradiction implies the statement of the theorem.  $\square$

Theorem 4.7 implies the following corollary:

**Corollary 4.8.** *A primitive Hausdorff semitopological inverse semigroup  $S$  is (absolutely)  $H$ -closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is every its maximal subgroup  $G$  with adjoined zero with an induced topology from  $S$ .*

**Remark 4.9.** We observe that the statements of Theorems 4.5, 4.6 and 4.7 hold for  $H$ -closed and absolute  $H$ -closed semitopological semilattices in the class of semitopological semilattices.

**Theorem 4.10.** *An infinite semitopological semigroup of  $\lambda \times \lambda$ -matrix units  $B_\lambda$  is  $H$ -closed in the class of semitopological semigroups if and only if the space  $B_\lambda$  is compact.*

*Proof.* Implication  $(\Leftarrow)$  is trivial.

$(\Rightarrow)$ . Suppose to the contrary that there exists a Hausdorff non-compact topology  $\tau_B$  on the semigroup  $B_\lambda$  such that  $(B_\lambda, \tau_B)$  is an  $H$ -closed semigroup in the class of semitopological semigroups. By Lemma 2 of [18] every non-zero element of  $B_\lambda$  is an isolated point in  $(B_\lambda, \tau_B)$ . Then there exists an infinite open-and-closed subset  $A \subseteq B_\lambda \setminus \{0\}$ .

Then we have that at least one of the following cases holds:



- 1) there exist finitely many  $i_1, \dots, i_n \in \lambda$  such that if  $(i, j) \in A$  then  $i \in \{i_1, \dots, i_n\}$ ;
- 2) there exist finitely many  $j_1, \dots, j_n \in \lambda$  such that if  $(i, j) \in A$  then  $i \in \{j_1, \dots, j_n\}$ ;
- 3) cases 1) and 2) don't hold.

Suppose case 1) holds. Then there exists an element  $i_0 \in \{i_1, \dots, i_n\}$  such that the set  $\{(i_0, j) : j \in \lambda\} \cap A$  is infinite. We denote  $A_{i_0} = \{(i_0, j) \in B_\lambda : (i_0, j) \in A\}$ . It is obvious that  $A_{i_0}$  is infinite subset of the semigroup  $B_\lambda$ . By Lemma 2 of [18] every non-zero element of  $B_\lambda$  is an isolated point in  $(B_\lambda, \tau_B)$  and hence  $A_{i_0}$  is an open-and-closed subset in the topological space  $(B_\lambda, \tau_B)$ . Since the left shift  $l_{(i_0, i)} : B_\lambda \rightarrow B_\lambda : x \mapsto (i_0, i) \cdot x$  is a continuous map for any  $i \in \lambda$ ,  $A_i = \{(i, j) \in B_\lambda : (i_0, j) \in A\}$  is an infinite open-and-closed subset in  $(B_\lambda, \tau_B)$  for every  $i \in \lambda$ . This implies that the set  $B_\lambda \setminus \{A_{i_1} \cup \dots \cup A_{i_k}\}$  is an open neighbourhood of the zero in  $(B_\lambda, \tau_B)$  for every finite subset  $\{i_1, \dots, i_k\} \subset \lambda$ .

Now, for every  $i \in \lambda$  we put  $a_i \notin B_\lambda$ . We extend the semigroup operation from  $B_\lambda$  onto the set  $S = B_\lambda \cup \{a_i : i \in \lambda\}$  in the following way:

- (i)  $a_i \cdot a_j = a_i \cdot 0 = 0 \cdot a_i = 0$  for all  $i, j \in \lambda$ ;
- (ii)  $(s, p) \cdot a_i = \begin{cases} a_s, & \text{if } p = i; \\ 0, & \text{if } p \neq i \end{cases}$  for all  $(s, p) \in B_\lambda \setminus \{0\}$  and  $i \in \lambda$ ;
- (iii)  $a_i \cdot (s, p) = 0$  for all  $(s, p) \in B_\lambda \setminus \{0\}$  and  $i \in \lambda$ .

Simple verifications show that so defines binary operation on  $S$  is associative, and hence  $S$  is a semigroup.

Next, we define a topology  $\tau_S$  on the semigroup  $S$  in the following way. For every element  $x \in B_\lambda$  we put that bases of topologies  $\tau_B$  and  $\tau_S$  at the point  $x$  coincide. Also, for every  $i \in \lambda$  we put

$$\mathcal{B}_S(a_i) = \{\{a_i\} \cup C_i : C_i \text{ is a cofinite subset of } A_i\}$$

is a base of the topology  $\tau_S$  at the point  $a_i \in S$ . It is obvious that  $(S, \tau_S)$  is a Hausdorff topological space. The separate continuity of the semigroup operation in  $(S, \tau_S)$  follows from the cofinality of the set  $C_i$  in  $A_i$  for each  $i \in \lambda$ . Therefore we get that the semitopological semigroup  $(B_\lambda, \tau_B)$  is a dense proper subsemigroup of  $(S, \tau_S)$ , which contradicts the assumption of the theorem.

In case 2) the proof is similar.

Suppose that cases 1) and 2) don't hold. By induction we construct an infinite sequence  $\{(x_i, y_i)\}_{i \in \mathbb{N}}$  in  $B_\lambda$  in the following way. First we fix an arbitrary element  $(x, y) \in A$  and denote  $(x_1, y_1) = (x, y)$ . Suppose that for some positive integer  $n$  we construct the finite sequence  $\{(x_i, y_i)\}_{i=1, \dots, n}$ . Since the set  $A$  is infinite and cases 1) and 2) don't hold, there exists  $(x, y) \in A$  such that  $x \notin \{y_1, \dots, y_n\}$  and  $y \notin \{x_1, \dots, x_n\}$ . Then we put  $(x_{n+1}, y_{n+1}) = (x, y)$ .

Let  $a \notin B_\lambda$ . We put  $T = B_\lambda \cup \{a\}$  and extend the semigroup operation from  $B_\lambda$  onto  $T$  in the following way:

$$a \cdot x = x \cdot a = a \cdot a = 0, \text{ for every } x \in B_\lambda.$$

Next, we define a topology  $\tau_T$  on the semigroup  $T$  in the following way. For every element  $x \in B_\lambda$  we put that bases of topologies  $\tau_B$  and  $\tau_T$  at the point  $x$  coincide. Also, we put

$$\mathcal{B}_T(a) = \{\{a\} \cup C : C \text{ is a cofinite subset of the set } \{(x_i, y_i) : i \in \mathbb{N}\}\}$$

is a base of the topology  $\tau_T$  at the point  $a \in T$ . It is obvious that  $(T, \tau_T)$  is a Hausdorff topological space, the semigroup operation in  $(T, \tau_T)$  is separately continuous, and  $B_\lambda$  is a dense subsemigroup of  $(T, \tau_T)$ . This contradicts the assumption of the theorem.

The obtained contradictions imply the statement of our theorem.  $\square$

**Remark 4.11.** By Theorem 2 [18] for every infinite cardinal  $\lambda$  there exists a unique Hausdorff pseudocompact topology  $\tau_c$  on the semigroup  $B_\lambda$  such that  $(B_\lambda, \tau_c)$  is a semitopological semigroup. This topology is compact and it is described in Example 1 of [18].

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